Linear Algebra and its Applications 435 (2011) 2259–2271

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Partial matrices whose completions have ranks bounded below

James McTigue, Rachel Quinlan *

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

ARTICLE INFO

Article history:
Received 10 February 2011
Accepted 12 April 2011
Available online 26 May 2011

Submitted by R.A. Brualdi

AMS classification:
15A03
15A83

Keywords:
Affine space
Partial matrix
Completion
Rank
Duality

ABSTRACT

A partial matrix over a field $\mathbb{F}$ is a matrix whose entries are either elements of $\mathbb{F}$ or independent indeterminates. A completion of such a partial matrix is obtained by specifying values from $\mathbb{F}$ for the indeterminates. We determine the maximum possible number of indeterminates in a partial $m \times n$ matrix whose completions all have rank at least equal to a particular k, and we fully describe those examples in which this maximum is attained. Our main theoretical tool, which is developed in Section 2, is a duality relationship between affine spaces of matrices in which ranks are bounded below and affine spaces of matrices in which the (left or right) nullspaces of elements possess a certain covering property.

1. Introduction

This article explores a relationship between two particular properties that may be possessed by affine spaces of matrices. The first of these properties is the condition that the ranks of elements in an affine subspace be bounded below. The second is the condition that an affine subspace, considered as a set of linear transformations operating on column (or row) vectors, should contain elements annihilating every small subspace. The two properties are possessed, respectively, by a pair of affine spaces that are related to each other by orthogonality with respect to the trace bilinear form. The meaning of "small" in the second property is a dimension bound depending on the greatest lower bound that applies in the first property.

* Corresponding author.
E-mail addresses: j.mctigue1@nuigalway.ie (J. McTigue), rachel.quinlan@nuigalway.ie (R. Quinlan).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2011.04.012

A special case of this duality was investigated by Quinlan [8], and used to obtain an upper bound for the dimension of a linear space of square matrices in which no element has a non-zero eigenvalue in the ground field. Linear spaces with this property are closely related to affine spaces of square matrices in which every element is non-singular. In Section 2 of the present article the context in which the duality is considered is extended to affine spaces of rectangular matrices in which the ranks that occur are bounded below. An upper bound for the dimension of such a space is obtained using the results of Quinlan [8] and a 1989 theorem of Meshulam [7]. In Section 3, the duality is applied to an investigation of partial matrices which have the maximum possible number of free entries, subject to the condition that their completions have ranks satisfying specified lower bounds. The main result of Section 3 is a complete description of such partial matrices over all fields, extending a recent theorem of Brualdi et al. [1] on square partial matrices whose completions are all non-singular.

Throughout this article $\mathbb{F}$ may be any field. The space of all $m \times n$ matrices with entries in $\mathbb{F}$ is denoted by $M_{m\times n}(\mathbb{F})$; this is contracted to $M_n(\mathbb{F})$ if $m = n$. The space of row vectors of length m with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^m$, and the space of column vectors by $(\mathbb{F}^m)^T$. The superscript T denotes transpose.

Definition 1.1. An *affine subspace* of $M_{m\times n}(\mathbb{F})$ is a coset of a linear subspace; it is a subset of $M_{m\times n}(\mathbb{F})$ of the form

$$C + \mathcal{S} = \{C + X : X \in \mathcal{S}\},$$

where C is a fixed element of $M_{m\times n}(\mathbb{F})$ and $\mathcal{S}$ is a linear subspace of $M_{m\times n}(\mathbb{F})$. The *dimension* of the affine subspace $C + \mathcal{S}$ is the dimension of the linear space $\mathcal{S}$.

Associated with a subspace $\mathcal{S}$ of $M_{m\times n}(\mathbb{F})$ is a subspace S^* of $M_{n\times m}(\mathbb{F})$ defined by

$$S^* = \{X \in M_{n\times m}(\mathbb{F}) : \mathrm{trace}(XY) = 0 \;\forall Y \in S\}.$$

Our main duality theorem, which appears in Section 2, may be stated as follows.

Theorem 2.9. Let $C + \mathcal{S}$ be an affine subspace of $M_{m\times n}(\mathbb{F})$ as in Definition 1.1, with $C \notin \mathcal{S}$. Let C' be an element of $\mathcal{S}^*$ for which $\mathrm{trace}(C'C) \neq 0$. Then every element of $C + \mathcal{S}$ has rank at least k if and only if every subspace of $(\mathbb{F}^m)^T$ of dimension $k - 1$ is contained in the right nullspace of some element of the affine space $C' + \langle C, \mathcal{S}\rangle^*$.

In Section 3, this duality theorem is applied to the problem of classifying partial matrices whose completions have ranks that are bounded below. A *partial matrix* over $\mathbb{F}$ is a matrix in which some entries are specified as elements of $\mathbb{F}$ and the remainder are independent indeterminates. A *completion* of such a partial matrix is the matrix resulting from a specific assignment of values in $\mathbb{F}$ to the indeterminate entries. The set of all completions of a partial matrix is an affine space, hence the connection to Theorem 2.9.

Problems involving ranks of completions of partial matrices have attracted considerable interest. Square partial matrices (over arbitrary fields) whose completions are all singular were first characterized by Hartfiel and Loewy [5]; the more general problem of classifying square partial matrices whose completions have ranks with a specified upper bound was solved in [2]. The same problems were studied by Brualdi et al. [1] in the more general framework of ACI-matrices. An ACI (*affine column independent*) matrix differs from a partial matrix in that its entries may be linear combinations of constants and indeterminates; however no indeterminate may appear in more than one column. When elementary row operations are applied to partial matrices, ACI-matrices arise in a natural way. Brualdi et al. obtain a characterization of ACI-matrices for which the ranks of all completions satisfy specified upper bounds, and use this to recover the solution for partial matrices. They also investigate $n \times n$ partial matrices whose completions are all non-singular. They identify the maximum possible number of indeterminates in such a matrix, and fully describe the examples in which this bound is attained. Their proofs of these latter results (Theorem 12 of [1]) require the hypothesis that the field order exceeds n. Huang and Zhan generalize this theorem in [6] by determining the maximum number of indeterminates in an $m \times n$ partial (or ACI) matrix having the property that all of its completions have the same specified rank, and by characterizing those examples in which this bound is attained, again subject to a condition on the field order.

Our results in Section 3 of this article may be regarded as extensions of Theorem 12 of [1] in a different direction. We give a new proof of this theorem that does not require any restriction on the field. Moreover, for any field $\mathbb{F}$, we determine the maximum possible number of indeterminates in a partial $m\times n$ matrix whose completions have ranks bounded below by a specified k, and we characterize those examples where this bound is attained. Our strategy is the application of Theorem 2.9. Rather than studying the rank bound directly, we investigate affine spaces having the dual property instead. These are particularly amenable to study in the special situation of affine spaces arising from completions of partial matrices, essentially because in this case the relevant property can be encapsulated in terms of rowspaces of completions having only a single non-zero row, as we discuss in Section 3.

It would be of interest to know whether the restriction on the field order can also be relaxed in the results of Huang and Zhan on partial matrices whose completions all have the same prescribed rank [6]. The techniques employed in our study do not directly address this situation.

2. Affine spaces with ranks bounded below, and a dual property

The theme of this section is a characterization of affine spaces of matrices in which the ranks of elements are bounded below, in terms of a dual property associated with the *trace bilinear form*.

Definition 2.1. For positive integers m and n, we define the *trace bilinear form* τ on $M_{m\times n}(\mathbb{F})$ by

$$\tau(A,B) = \text{trace}(A^TB),$$

for $A,B \in M_{m\times n}(\mathbb{F})$.

Then τ is a nondegenerate symmetric bilinear form on $M_{m\times n}(\mathbb{F})$. For a linear subspace S of $M_{m\times n}(\mathbb{F})$, we use the term *trace complement* of S to refer to the linear subspace of $M_{n\times m}(\mathbb{F})$ consisting of all those matrices $X \in M_{n\times m}(\mathbb{F})$ for which $\text{trace}(XY) = 0$ for all $Y \in S$. Thus the trace complement of S, which we denote by S^*, is the transpose of the orthogonal complement of S with respect to the trace bilinear form τ. For an element A of $M_{m\times n}(\mathbb{F})$, the notation A^* will be used for the trace complement of the one-dimensional space $\langle A\rangle$. It follows from standard properties of nondegenerate symmetric bilinear forms that for any subspace S of $M_{m\times n}(\mathbb{F})$, $\dim S + \dim S^* = mn$. In the case $n = 1$, the trace complement of a subspace of $\mathbb{F}^m$ is just the transpose of the orthogonal complement with respect to the usual scalar product.

We begin by considering the special situation of an affine space of square matrices in which every element is non-singular. Let R_1 be a linear subspace of $M_n(\mathbb{F})$ and let B be an element of $M_n(\mathbb{F})$ for which every element of the affine space $B + R_1$ is non-singular. Then $B \notin R_1$ and if R denotes the linear subspace of $M_n(\mathbb{F})$ spanned by B and R_1, every element of the set $R\backslash R_1$ is non-singular. Write $S_1 = R_1^*$ and $S = R^*$. Then S has codimension 1 in S_1, and our first theorem asserts that the non-singularity of all elements of $R\backslash R_1$ is equivalent to the statement that $S_1\backslash S$ has the *hyperplane annihilation property* on $(\mathbb{F}^n)^T$, as defined below.

Definition 2.2. Let $\mathcal{S}$ be a subset of $M_{n\times m}(\mathbb{F})$. Then $\mathcal{S}$ has the *hyperplane annihilation property* on $(\mathbb{F}^m)^T$ if every hyperplane H of $(\mathbb{F}^m)^T$ is contained in the right nullspace of some element of $\mathcal{S}$.

In our usual contexts, the zero matrix will not belong to sets in which the hyperplane annihilation property and related properties are of interest. If $0 \notin \mathcal{S} \subseteq M_{n\times m}(\mathbb{F})$, the statement that $\mathcal{S}$ has the hyperplane annihilation property on $(\mathbb{F}^m)^T$ means that for every hyperplane H of $(\mathbb{F}^m)^T$, some element of $\mathcal{S}$ has the one-dimensional space H^* as its rowspace. Equivalently, in this situation every non-zero element of $\mathbb{F}^m$ spans the rowspace of some element of $\mathcal{S}$ of rank 1. This is a useful interpretation of the hyperplane annihilation property for our purposes.

Theorem 2.3. *Let R be a linear subspace of $M_n(\mathbb{F})$ and let R_1 be a subspace of R of codimension 1. Let $S_1 = R_1^*$ and let $S = R^*$. Then every element of $R \backslash R_1$ is non-singular if and only if $S_1 \backslash S$ has the hyperplane annihilation property on $(\mathbb{F}^n)^T$.*

Proof. Suppose first that every element of $R \backslash R_1$ is non-singular, and let $B \in R \backslash R_1$. Since R_1 has codimension 1 in R, an element A of $M_n(\mathbb{F})$ belongs to $S_1 \backslash S$ if $\text{trace}(AX) = 0$ for all $X \in R_1$ and $\text{trace}(AB) \neq 0$. Let v be a non-zero element of $\mathbb{F}^n$. We need to show that some element of $S_1 \backslash S$ has rowspace $\langle v \rangle$.

Since every element of $B + R_1$ is invertible, the product $v(B + X)$ is not the zero vector for any $X \in R_1$. Thus the row vector vX is not equal to $-vB$ for any $X \in R_1$, and vB does not belong to the subspace $\{vX : X \in R_1\}$. This means $\{vX : X \in R_1\}^* \not\subseteq (vB)^*$ in $(\mathbb{F}^n)^T$, and so there is an element u of $\mathbb{F}^n$ for which

$$(vX)u^T = 0 \; \forall \, X \in R_1, \text{ and } (vB)u^T \neq 0.$$

Now

$$(vX)u^T = \text{trace}(u^T(vX)) = \text{trace}((u^T v)X) = 0$$

for all $X \in R_1$, and

$$(vB)u^T = \text{trace}(u^T(vB)) = \text{trace}((u^T v)B) \neq 0.$$

Thus the $n \times n$ matrix $u^T v$ belongs to $R_1^* = S_1$ but not to $R^* = S$, and its rowspace is $\langle v \rangle$. So $S_1 \backslash S$ has the hyperplane annihilation property on $(\mathbb{F}^n)^T$.

On the other hand suppose that $S_1 \backslash S$ has the hyperplane annihilation property on $(\mathbb{F}^n)^T$, and let $B' \in R \backslash R_1$. Choose any non-zero $v \in \mathbb{F}^n$, and let A_v be an element of $S_1 \backslash S$ whose rowspace is $\langle v \rangle$. Then the subspace A_v^* of $M_n(\mathbb{F})$ contains R_1 but not R. Since R_1 has codimension 1 in R this means that A_v^* intersects R exactly in R_1, hence $B' \notin A_v^*$. Thus $\text{trace}(A_v B') \neq 0$, and this means that there is at least one column of B' that is not in the trace complement of the row vector v. Since this is true for every non-zero $v \in \mathbb{F}^n$, it follows that the columns of B' span $(\mathbb{F}^n)^T$. Thus B' is non-singular. $\square$

Suppose as in the proof of Theorem 2.3 that $B + R_1$ is an affine subspace of $M_n(\mathbb{F})$ in which every element is non-singular. Then so also is $B^{-1}(B + R_1) = I_n + B^{-1}R_1$. Write R_1' for the linear space $B^{-1}R_1$. The statement that every element of $I_n + R_1'$ is invertible means precisely that no element of R_1' possesses a non-zero eigenvalue that belongs to the field $\mathbb{F}$. Write S_1' for the trace complement of R_1' in $M_n(\mathbb{F})$. The trace complement in $M_n(\mathbb{F})$ of the identity matrix I_n is $\mathcal{T}$, the space of matrices of trace zero in $M_n(\mathbb{F})$. Thus Theorem 2.3 implies that $S_1' \backslash S_1' \cap \mathcal{T}$ has the hyperplane annihilation property on $(\mathbb{F}^n)^T$ if and only if no element of R_1' possesses a non-zero eigenvalue in $\mathbb{F}$. This special case of Theorem 2.3 is discussed in [8], where the following bound for the dimension of S_1' is established using an induction argument. An essential component of that argument is a demonstration that if S is a subspace of $M_n(\mathbb{F})$ and $S \backslash S \cap \mathcal{T}$ has the hyperplane annihilation property on $(\mathbb{F}^n)^T$, then there is an element v of $(\mathbb{F}^n)^T$ for which $\{Av : A \in S\} = (\mathbb{F}^n)^T$. A related observation will be relevant later in this article in Lemma 3.8.

Theorem 2.4. *For any field $\mathbb{F}$, the minimum possible dimension of a linear subspace S of $M_n(\mathbb{F})$ for which $S \backslash S \cap \mathcal{T}$ has the hyperplane annihilation property on $(\mathbb{F}^n)^T$ is $\frac{n(n+1)}{2}$.*

Examples of spaces satisfying the condition and attaining the dimension bound of Theorem 2.4 include the space of upper triangular matrices in $M_n(\mathbb{F})$ for any field, and the space of symmetric matrices in $M_n(\mathbb{F})$ if $\mathbb{F}$ is a formally real field.

The following statement, which by Theorem 2.3 is equivalent to Theorem 2.4, is proved in [8] and independently (using a different approach) by de Seguins Pazzis [3].

Theorem 2.5. *For any field $\mathbb{F}$, a subspace of $M_n(\mathbb{F})$ in which no element has a non-zero eigenvalue belonging to $\mathbb{F}$ may have dimension at most $\frac{n(n-1)}{2}$.*

Examples of spaces satisfying the condition and dimension bound of Theorem 2.5 include the space of strictly upper triangular $n \times n$ matrices over any field, and the space of skew-symmetric $n \times n$ matrices over a formally real field. The following immediate consequence of Theorem 2.5 appears in [3].

Corollary 2.6. *For any field $\mathbb{F}$, the maximum possible dimension of an affine subspace of $M_n(\mathbb{F})$ in which every element is non-singular is $\frac{n(n-1)}{2}$.*

We also note that $\frac{n(n+1)}{2}$ is the minimum possible dimension of a linear subspace S_1 of $M_n(\mathbb{F})$ that contains a subspace S of codimension 1 for which $S_1 \backslash S$ has the hyperplane annihilation property on $(\mathbb{F}^n)^T$. This observation is equivalent to Corollary 2.6, by Theorem 2.3.

It was shown by Meshulam [7] that if $\mathbb{K}$ is an algebraically closed field or if $\mathbb{K} = \mathbb{R}$, then the maximum possible dimension of an affine subspace of $M_{m\times n}(\mathbb{K})$ in which every element has rank at least k, for some fixed $k \leqslant \min(m, n)$, is $mn - \frac{k(k+1)}{2}$. The only property of algebraically closed fields and of $\mathbb{R}$ that is used in Meshulam's proof is the fact that Theorem 2.5 holds for these fields. For the field $\mathbb{R}$ of real numbers, Theorem 2.5 is an immediate consequence of the observation that any linear subspace of $M_n(\mathbb{R})$ whose dimension exceeds $\frac{n(n-1)}{2}$ must intersect the space of symmetric matrices non-trivially, and every non-zero real symmetric matrix has a non-zero real eigenvalue. For an algebraically closed field $\mathbb{K}$, Theorem 2.5 amounts to the statement that a linear subspace of $M_n(\mathbb{K})$ in which every element is nilpotent can have dimension at most $\frac{n(n-1)}{2}$. This was originally proved (for any field with at least n elements) by Gerstenhaber in 1958 [4].

Since Theorem 2.5 holds for all fields, Meshulam's proof of the dimension bound for affine spaces in which ranks are bounded below extends unproblematically to all fields. This proof is included below for completeness.

Theorem 2.7. *Let k, m and n be positive integers with $k \leqslant \min(m, n)$. Suppose that R is a subspace of $M_{m\times n}(\mathbb{F})$, and that R_1 is a subspace of R of codimension 1 for which every element of $R\backslash R_1$ has rank at least k. Then the dimension of R_1 is at most equal to $mn - \frac{k(k+1)}{2}$.*

Proof. Suppose that $l \geqslant k$ is the least rank that occurs in $R\backslash R_1$, and let B be an element of $R\backslash R_1$ of rank l. Then there exist non-singular matrices $P \in \mathrm{GL}(m, \mathbb{F})$ and $Q \in \mathrm{GL}(n, \mathbb{F})$ for which the $m \times n$ matrix PBQ has a copy of the $l \times l$ identity matrix I_l in its upper left $l \times l$ region and is otherwise full of zeroes. Write R' and R'_1, respectively, for the spaces PRQ and PR_1Q; these spaces have the same respective dimensions as R and R_1, and every element of $R'\backslash R'_1$ has rank at least l.

Now let R_l be the subspace of R' consisting of those elements whose non-zero entries are all located in the upper left $l \times l$ region. Since l is the least rank of any element in $R'\backslash R'_1$, every element of $R_l\backslash R_l \cap R'_1$ has rank exactly l and so $\dim(R_l \cap R'_1) \leqslant \frac{l(l-1)}{2}$ by Corollary 2.6; equivalently $\dim R_l \leqslant \frac{l(l-1)}{2} + 1$.

Now a complement of R_l in R' can have dimension at most $mn - l^2$, since its projection onto the subspace of $M_{m\times n}(\mathbb{F})$ consisting of elements with zeroes in the upper left $l \times l$ region must have trivial kernel. Thus $\dim R' \leqslant \frac{l(l-1)}{2} + 1 + mn - l^2$ and

$$\dim R'_1 \leqslant mn - \frac{l(l+1)}{2}.$$

Since $\dim R'_1 = \dim R_1$ and $l \geqslant k$, we conclude that

$$\dim R_1 \leqslant mn - \frac{k(k+1)}{2},$$

and that this bound can be attained only if $R\backslash R_1$ contains an element of rank k. $\square$

The duality considered in Theorem 2.3 involved a connection between affine subspaces of $M_n(\mathbb{F})$ consisting fully of non-singular elements, and affine subspaces of $M_n(\mathbb{F})$ possessing the *hyperplane annihilation property*. The remainder of this section consists of a discussion of an extension of Theorem 2.3 to the case of an affine subspace of $M_{m\times n}(\mathbb{F})$ in which the ranks that appear are bounded below as in the situation of Theorem 2.7. In order to state the duality theorem we need to generalize the notion of the hyperplane annihilation property.

Definition 2.8. A subset $\mathcal{S}$ of $M_{n\times m}(\mathbb{F})$ has the *dimension k annihilation property* on $(\mathbb{F}^m)^T$ if every k-dimensional subspace of $(\mathbb{F}^m)^T$ is contained in the right nullspace of some element of $\mathcal{S}$.

Thus $\mathcal{S}$ has the dimension k annihilation property on $(\mathbb{F}^m)^T$ if for every subspace V of $\mathbb{F}^m$ of dimension $m-k$, some element of $\mathcal{S}$ has a subspace of V as its rowspace. If $0 \in \mathcal{S} \subseteq M_{n\times m}(\mathbb{F})$ then $\mathcal{S}$ obviously has the dimension k annihilation property for all $k \leqslant m$; in the cases of interest $\mathcal{S}$ will typically be (related to) an affine subspace of $M_{n\times m}(\mathbb{F})$ and will not include the zero matrix.

The following is our main duality theorem. Its proof is an adaptation of our earlier proof of the special case Theorem 2.3, and it uses the following notation.

For a matrix $M \in M_{m\times n}(\mathbb{F})$ and a positive integer r, we let $M^{\oplus r}$ denote the $rm \times rn$ matrix that is defined by

$$(M^{\oplus r})_{tm+i,tn+j} = M_{ij} \quad \text{for } 0 \leqslant t \leqslant r-1, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n,\ (M^{\oplus r})_{p,q} = 0 \text{ otherwise.}$$

Thus $M^{\oplus r}$ has r appearances of M as blocks arranged diagonally from upper left to lower right, and zeroes elsewhere.

Theorem 2.9. *Let k, m and n be positive integers with $k \leqslant \min(m,n)$. Suppose that R is a subspace of $M_{m\times n}(\mathbb{F})$ and that R_1 is a subspace of R of codimension 1. Let S_1 and S, respectively, denote the trace complements of R_1 and R in $M_{n\times m}(\mathbb{F})$. Then every element of $R\backslash R_1$ has rank at least k if and only if $S_1\backslash S$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$.*

Proof. First suppose that every element of $R\backslash R_1$ has rank at least k. Let V be a subspace of $\mathbb{F}^m$ of dimension $m-k+1$, and let $\{v_1, v_2, \ldots, v_{m-k+1}\}$ be a basis of V. We require to show that some element of $S_1\backslash S$ has a subspace of V as its rowspace.

Let $B \in R\backslash R_1$. Every element of $B+R_1$ has rank at least k, so there is no $X \in R_1$ for which $v_i(B+X) = 0_{1\times n}$ for every i in the range $1, \ldots, m-k+1$. Thus there is no $X \in S$ for which

$$v_iX = -v_iB, \quad \forall i \in \{1, \ldots, m-k+1\}.$$

Let v be the element of $\mathbb{F}^{m(m-k+1)}$ obtained by concatenating $v_1, v_2, \ldots, v_{m-k+1}$. The vector $vB^{\oplus m-k+1}$ is the vector in $\mathbb{F}^{n(m-k+1)}$ obtained by concatenating $v_1B,\ v_2B, \ldots, v_{m-k+1}B$. This vector does not belong to the subspace

$$\{vX^{\oplus m-k+1} : X \in R_1\}$$

of $\mathbb{F}^{n(m-k+1)}$. This means that $\{vX^{\oplus m-k+1} : X \in R_1\}^* \not\subseteq (vB^{\oplus m-k+1})^*$ in $(\mathbb{F}^{n(m-k+1)})^T$. Hence there exists an element u of $\mathbb{F}^{n(m-k+1)}$ for which

$$vX^{\oplus m-k+1}\, u^T = 0\ \forall\, X \in R_1 \quad \text{and} \quad vB^{\oplus m-k+1}\, u^T \neq 0.$$

Now for $i = 1, \ldots, m-k+1$, let u_i be the element of $\mathbb{F}^n$ consisting of the entries in positions $n(i-1)+1$ through ni of v : thus the vector $u \in \mathbb{F}^{n(m-k+1)}$ is the concatenation of the vectors $u_1, \ldots, u_{m-k+1}$ in $\mathbb{F}^n$.

If M is any matrix in $M_{m\times n}(\mathbb{F})$, then $u^T\, vM^{\oplus m-k+1}$ is a square matrix in $M_{n(m-k+1)}(\mathbb{F})$ that has $u_i^T\, v_iM$ as its ith diagonal $n \times n$ block. Thus

$$\operatorname{trace}(u^T vM^{\oplus m-k+1}) = \sum_{i=1}^{m-k+1} \operatorname{trace}\left(u_i^T v_i M\right)$$
$$= \operatorname{trace}\left(\sum_{i=1}^{m-k+1} u_i^T v_i M\right).$$

Now the rowspace of the $n \times m$ matrix $\sum_{i=1}^{m-k+1} u_i^T v_i$ is a subspace of V, and this matrix has the following properties :
If $X \in R_1$, then

$$\operatorname{trace}\left(\left(\sum_{i=1}^{m-k+1} u_i^T v_i\right) X\right) = \operatorname{trace}(u^T vX^{\oplus m-k+1}) = vX^{\oplus m-k+1} u^T = 0$$

and

$$\operatorname{trace}\left(\left(\sum_{i=1}^{m-k+1} u_i^T v_i\right) B\right) = \operatorname{trace}(u^T vB^{\oplus m-k+1}) = vB^{\oplus m-k+1} u^T \neq 0.$$

Thus $\sum_{i=1}^{m-k+1} u_i^T v_i \in S_1 \backslash S$. Since the rowspace of this matrix is a subspace of V, we have shown that every subspace of $\mathbb{F}^m$ of dimension $m-(k-1)$ contains the rowspace of some element of $S_1 \backslash S$, hence that $S_1 \backslash S$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$, as required.

On the other hand suppose that $S_1 \backslash S$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$, and let $B' \in R \backslash R_1$. Let V' be a subspace of $\mathbb{F}^m$ of dimension $m-k+1$, and let $A_{V'}$ be an element of $S_1 \backslash S$ whose rowspace is a subspace of V'. Then $\operatorname{trace}(A_{V'} B') \neq 0$, and it follows that there is an element v' of V' for which $v'B' \neq 0$. Thus the left nullspace of B' contains no subspace of $\mathbb{F}^m$ of dimension exceeding $m-k$, so the rank of B' is at least k. $\square$

The following extension of Theorem 2.4 is immediate from Theorems 2.7 and 2.9.

Theorem 2.10. *Let $k \in \{1, \ldots, n\}$. Let S_1 be a subspace of $M_{n\times m}(\mathbb{F})$, and let S be a subspace of S_1 of codimension 1 for which $S_1 \backslash S$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$. Then*

$$\dim S_1 \geqslant \frac{k(k+1)}{2}.$$

Examples of subspaces of $M_{n\times m}(\mathbb{F})$ realizing the conditions and dimension bound of Theorem 2.10 may be constructed as follows. Let S_1' be a subspace of $M_k(\mathbb{F})$ of dimension $\frac{k(k+1)}{2}$, containing a subspace S' of codimension 1 for which $S_1' \backslash S'$ has the hyperplane annihilation property on $(\mathbb{F}^k)^T$. (For example S_1' could be the space of all upper triangular matrices in $M_k(\mathbb{F})$, with S' being the subspace of S_1' consisting of all elements of trace zero.) Let S_1 and S be the subspaces of $M_{n\times m}(\mathbb{F})$ consisting of all matrices that have an element of S_1', respectively S', in the upper left $k \times k$ region and are otherwise filled with zeroes. Then $S_1 \backslash S$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$. To see this, let U be a subspace of $(\mathbb{F}^m)^T$ of dimension $k-1$. The trace complement in $\mathbb{F}^m$ of U has dimension $m-k+1$ and therefore intersects every k-dimensional subspace of $\mathbb{F}^m$ non-trivially. Thus every U is annihilated by some non-zero element v of $\mathbb{F}^m$ whose non-zero entries are all in the first k positions, and v spans the rowspace of some element of $S_1' \backslash S'$.

We conclude Section 2 now by describing the preservation under transposition of our annihilation properties, for sets of matrices of relevant types. Lemma 2.11 below will be used in Section 3, when we apply the relationship between rank bounds and annihilation properties to the problem of classifying partial matrices whose completions satisfy certain rank conditions. For a subset $\mathcal{S}$ of $M_{n\times m}(\mathbb{F})$, we let $\mathcal{S}^T$ denote the set of transposes of elements of $\mathcal{S}$ in $M_{m\times n}(\mathbb{F})$.

Lemma 2.11. *Suppose that $S_1\backslash S$ has the dimension k annihilation property on $(\mathbb{F}^m)^T$ for some $k<m$, for a subspace S_1 of $M_{n\times m}(\mathbb{F})$ and a subspace S of codimension 1 in S_1. Then the subset $S_1^T\backslash S^T$ of $M_{m\times n}(\mathbb{F})$ has the dimension k annihilation property on $(\mathbb{F}^n)^T$.*

Proof. Let $R=S^*$ and let $R_1=S_1^*$ in $M_{m\times n}(\mathbb{F})$. Then, by Theorem 2.9, every element of $R\backslash R_1$ has rank at least $k+1$. Thus every element of the subset $R^T\backslash R_1^T$ of $M_{n\times m}(\mathbb{F})$ has rank at least $k+1$ and, by Theorem 2.9 again, $(R_1^T)^*\backslash(R^T)^*$ has the dimension k annihilation property on $(\mathbb{F}^n)^T$.

Now

$$\begin{aligned}
(R^T)^* &= \{X\in M_{m\times n}(\mathbb{F}) : \mathrm{trace}(XY^T)=0,\ \forall\, Y\in R\}\\
&= \{X\in M_{m\times n}(\mathbb{F}) : \mathrm{trace}(YX^T)=0,\ \forall\, Y\in R\}\\
&= \{X\in M_{m\times n}(\mathbb{F}) : X^T\in R^*\}\\
&= (R^*)^T\\
&= S^T.
\end{aligned}$$

Similarly $(R_1^T)^*=(S_1)^T$, and so $S_1^T\backslash S^T$ has the dimension k annihilation property on $(\mathbb{F}^n)^T$. $\square$

Lemma 2.11 can be interpreted as saying that our characterization of the dimension k annihilation property in terms of rowspaces of elements of $S\backslash S_1$ may equally well be expressed in terms of columns. Indeed this observation is already implicit in Theorem 2.9 and its proof, since the arguments there could be framed in terms of columns instead of rows.

3. Completions of partial matrices

A *partial matrix* over the field $\mathbb{F}$ is a matrix in which some entries are specified elements of $\mathbb{F}$ and the remainder are independent indeterminates. For positive integers m and n, we write $P_{m\times n}(\mathbb{F})$ for the set of all partial $m\times n$ matrices over $\mathbb{F}$, and abbreviate this to $P_n(\mathbb{F})$ if $m=n$.

An $\mathbb{F}$-*completion* (or simply *completion*) of a partial matrix in $P_{m\times n}(\mathbb{F})$ is the element of $M_{m\times n}(\mathbb{F})$ that results from an assignment of values in $\mathbb{F}$ to the indeterminate entries. The set $\mathrm{Comp}(A)$ of all completions of a partial matrix $A\in P_{m\times n}(\mathbb{F})$ is an affine subspace of $M_{m\times n}(\mathbb{F})$ whose dimension is equal to the number of indeterminates in A. This affine space is a linear space precisely if all the constant entries of A are zeroes.

The following terminology will be convenient. For a partial matrix $A\in P_{m\times n}(\mathbb{F})$, let C denote the element of $M_{m\times n}(\mathbb{F})$ obtained by assigning the value 0 to all the indeterminate entries of A. Let X denote the partial matrix that has zeroes in all positions occupied by constants in A and coincides with A in the positions occupied by indeterminates. We refer to C and X, respectively, as the *constant part* and *indeterminate part* of A. Then $A=C+X$, and if $\mathcal{X}$ denotes the linear subspace of $M_{m\times n}(\mathbb{F})$ consisting of all completions of X, then $\mathrm{Comp}(A)=C+\mathcal{X}$ as an affine subspace of $M_{m\times n}(\mathbb{F})$.

The following theorem on square partial matrices whose completions are all non-singular is proved by Brualdi et al. [1].

Theorem 3.1. *Let $\mathbb{F}$ be a field with at least $n+1$ elements, and let $A\in P_n(\mathbb{F})$ be a partial matrix all of whose completions have rank n. Then the number of indeterminates in A is at most equal to $\frac{n(n-1)}{2}$. This bound is attained if and only if there exist row and column permutations that transform A to an upper triangular partial matrix having non-zero constant entries on the main diagonal, and independent indeterminates above the main diagonal.*

Our goal in this section is to use the duality considerations of Section 2 to prove the following extension of Theorem 3.1. We describe those partial matrices for which the ranks of completions have a specified lower bound, and which have the maximum possible number of indeterminates subject to this property. Our theorem and proof apply uniformly for all fields.

Theorem 3.2. *Let $\mathbb{F}$ be a field and let $A \in P_{m\times n}(\mathbb{F})$ be a partial matrix with the property that every $\mathbb{F}$-completion of A has rank at least k for some fixed $k \leqslant \min(m,n)$. Then the number of indeterminates in A is at most $mn - \frac{k(k+1)}{2}$. This bound is attained if and only if A may be transformed by row and column permutations to a partial matrix A' of the following form:*

- *The upper left $k \times k$ submatrix of A' is an upper triangular matrix with non-zero constant entries on the main diagonal and independent indeterminates above the main diagonal.*
- *All entries of A' outside the upper left $k \times k$ region are independent indeterminates.*

The upper bound of Theorem 3.2 for the number of indeterminates in A is an immediate consequence of Theorem 2.7; what needs to be established is the assertion that all examples in which this bound is attained have the stated form. We do this using the machinery that was developed in Section 2. A key ingredient is the case $k = \min(m,n)$ and we begin by considering this situation, essentially presenting a new proof of Theorem 12 of [1] (Theorem 3.1 above) under more general hypotheses.

There is no loss of generality in assuming that $m \leqslant n$, and we make this assumption throughout the remainder of this article.

Theorem 3.3. *Let A be a partial matrix in $P_{m\times n}(\mathbb{F})$ having $mn - \frac{m(m+1)}{2}$ indeterminates and having the property that every element of $\mathrm{Comp}(A)$ has rank m. Then there exist permutation matrices $P \in M_m(\mathbb{F})$ and $Q \in M_n(\mathbb{F})$ for which the partial matrix $A' = PAQ$ has the following form :*

- *The $m \times m$ matrix consisting of the first m columns of A' is upper triangular, with non-zero constants on the main diagonal and independent indeterminates above the main diagonal.*
- *Any further columns of A' are fully occupied by independent indeterminates.*

The following elementary lemma will be needed for the proof of Theorem 3.3.

Lemma 3.4. *Let U be a subspace of $\mathbb{F}^m$ of dimension r, where $1 \leqslant r \leqslant m-1$. Then U contains an element with at least r non-zero entries.*

Proof. The subspace U of $\mathbb{F}^m$ may be represented as the left nullspace of a matrix in $M_{m\times(m-r)}(\mathbb{F})$ that has rank $m-r$ and is in reduced column-echelon form. Thus in order to specify an element of U, we have a free choice for the entries in r positions. $\square$

Our proof of Theorem 3.3 is presented as a series of three propositions, for the sake of clarity and comprehensibility. We begin by writing C and X, respectively, for the constant and indeterminate parts of A, and by writing $\mathcal{X}$ for the linear space $\mathrm{Comp}(X)$, which has dimension $mn - \frac{m(m+1)}{2}$. Write $\mathcal{Y}$ for the trace complement of $\mathcal{X}$, which is a linear subspace of $M_{n\times m}(\mathbb{F})$ of dimension $\frac{m(m+1)}{2}$. Note that $\mathcal{Y} = \mathrm{Comp}(Y)$, where Y is the partial $n \times m$ matrix having independent indeterminates precisely in the positions occupied by zeroes in X^T, and having zeroes in the positions occupied by indeterminates in X^T. Our first step is concerned with the distribution of the $m(m+1)/2$ indeterminates among the rows of Y.

Proposition 3.5. *There exists a permutation matrix $P \in M_n(\mathbb{F})$ for which the partial matrix $PY \in P_{n\times m}(\mathbb{F})$ has $m-i+1$ indeterminates in Row i for $i = 1,\dots,m$ and has only zeroes outside the first m rows.*

Proof. The affine space $C + \mathcal{X}$ has the property that all of its elements have rank m. Thus it follows from Theorem 2.9 that the subset $\mathcal{Y}\backslash\mathcal{Y}\cap C^*$ of $M_{n\times m}(\mathbb{F})$ has the hyperplane annihilation property on

$(\mathbb{F}^m)^T$. This means that every subspace of $(\mathbb{F}^m)^T$ of dimension $m-1$ is annihilated by some element of $\mathcal{Y}\backslash\mathcal{Y}\cap C^*$, or equivalently that every non-zero element of $\mathbb{F}^m$ occurs as a basis of the rowspace of an element of $\mathcal{Y}\backslash\mathcal{Y}\cap C^*$ of rank 1. In fact, in view of the special structure of the subspace $\mathcal{Y}$ of $M_{n\times m}(\mathbb{F})$, we may interpret the statement that $\mathcal{Y}\backslash\mathcal{Y}\cap C^*$ has the hyperplane annihilation property on $\mathbb{F}^m$ to mean that every non-zero element of $\mathbb{F}^m$ occurs as the unique non-zero row of some element of $\mathcal{Y}\backslash\mathcal{Y}\cap C^*$.

In particular every element of $\mathbb{F}^m$ that has no non-zero entries must occur as a row in some completion of Y, and so there is some $j_1 \in \{1,\ldots,n\}$ for which Row j_1 of Y contains m indeterminates. There exists a hyperplane H_1 of $\mathbb{F}^m$ consisting of elements that are in the trace complement of Column j_1 of C. If v is a non-zero element of H_1, then v must span the rowspace of some completion of Y that has non-zero entries outside Row j_1. Since H_1 has elements with at least $m-1$ non-zero entries by Lemma 3.4, there is some $j_2 \neq j_1$ for which Row j_2 of Y has at least $m-1$ indeterminates. Now the trace complements of Columns j_1 and j_2 of C intersect in a subspace H_2 of $\mathbb{F}^m$ of dimension at least $m-2$. By Lemma 3.4, H_2 contains an element with at least $m-2$ non-zero entries, hence there exists some $j_3 \notin \{j_1,j_2\}$ for which Row j_3 of Y contains at least $m-2$ indeterminates.

This iterative argument concludes with a list $j_1,j_2,\ldots,j_m$ of indices of distinct rows of Y, with the property that for $i=1,\ldots,m$, Row j_i of Y contains at least $m-i+1$ indeterminates. Since the total number of indeterminates in Y is $\frac{m(m+1)}{2}$, it must be that Row j_i of Y contains exactly $m-i+1$ indeterminates for $i=1,\ldots,m$, and that all entries of Y outside these m rows are zeroes.

We reach the desired conclusion by permuting the rows of Y so that the first m rows are the non-zero rows, and they are arranged in decreasing order of number of indeterminates. □

Note that in the situation of Proposition 3.5, $P\mathcal{Y}$ is the trace complement of $\mathcal{X}P^{-1}$, and that $\mathcal{X}P^{-1}$ is the linear space of completions of XP^{-1}. This matrix has independent indeterminates in the positions where $(PY)^T$ has zeroes, and has zeroes elsewhere. Thus the number of indeterminates in Column j of XP^{-1}, or equivalently of AP^{-1}, is $j-1$ if $j \leqslant m$, and is m if $n > m$ and $j > m$. We have shown that the partial matrix A may be transformed by a column permutation to one with this pattern of indeterminate positions. Now $CP^{-1}+\mathcal{X}P^{-1}$ is an affine subspace of $M_{m\times n}(\mathbb{F})$ in which every element has rank m, so $P\mathcal{Y}\backslash P\mathcal{Y}\cap(CP^{-1})^*$ is a subset of $M_{n\times m}(\mathbb{F})$ that has the hyperplane annihilation property on $(\mathbb{F}^m)^T$. Note that all non-zero entries of CP^{-1} occur in the first m columns.

The next step in our proof is the application of Lemma 2.11 to the concluding position of Proposition 3.5, to describe the distribution of indeterminates amongst the columns of PY.

Proposition 3.6. *There exists a permutation matrix $Q \in M_m(\mathbb{F})$ for which the entry in the (i,j) position of PYQ is an indeterminate if $j \geqslant i$ and is zero otherwise.*

Proof. Since $P\mathcal{Y}\backslash P\mathcal{Y}\cap(CP^{-1})^*$ has the dimension $(m-1)$ annihilation property on $(\mathbb{F}^m)^T$, it follows from Lemma 2.11 that the subset $(P\mathcal{Y})^T\backslash(P\mathcal{Y})^T\cap((CP^{-1})^T)^*$ of $M_{m\times n}(\mathbb{F})$ has the dimension $(m-1)$ annihilation property on $(\mathbb{F}^n)^T$. This means that every subspace of $(\mathbb{F}^n)^T$ of dimension $m-1$ is annihilated by some completion of $(PY)^T$ that is not in the trace complement of $(CP^{-1})^T$.

All entries of $(PY)^T$ outside the first m columns are zeroes. Let $\mathbb{F}^n_m$ denote the m-dimensional subspace of $\mathbb{F}^n$ consisting of all elements having zeroes outside the first m positions. The dimension $(m-1)$ annihilation property on $(\mathbb{F}^n)^T$ implies the same property on $(\mathbb{F}^n_m)^T$, and it follows that every element of $\mathbb{F}^n_m$ must span the rowspace of a completion of $(PY)^T$ of rank 1 that is not in the trace complement of $(CP^{-1})^T$. Thus we are essentially back in the situation of Proposition 3.5, and the argument employed there can be used to show that the indeterminates of $(PY)^T$ are confined to m rows, of which one contains m indeterminates, another contains $m-1$ indeterminates, and so on.

Thus the numbers of indeterminates in the m columns of PY are $1,2,\ldots,m$, in some order. Then there exists a permutation matrix $Q \in M_m(\mathbb{F})$ for which the columns of PYQ are arranged in increasing order of number of indeterminates. Since Row i of PY, hence also of PYQ, has exactly $m-i+1$ indeterminates for $i=1,\ldots,m$, it follows that PYQ has the required form. □

Now $P\mathcal{Y}Q$ is the trace complement of $Q^{-1}\mathcal{X}P^{-1}$, which is the indeterminate part of the partial matrix A' obtained by applying the permutations represented by Q^{-1} and P^{-1}, respectively, to the rows and columns of A. Let C' denote the constant part of A'. Since the $mn-\frac{m(m+1)}{2}$ indeterminates of A' occupy all the positions (i,j) with $i < j$, the non-zero entries of C' must all be located in those positions (i,j) with $i \geqslant j$. The final step in our proof of Theorem 3.3 is to identify the positions of the non-zero entries of C'.

Proposition 3.7. *The constant part C' of A' has non-zero entries precisely in the positions (i,i), for $i = 1,\ldots,m$.*

Proof. As observed above, that C' can have non-zero entries only in positions (i,j) with $i \geqslant j$ is apparent from the locations of the indeterminate entries of A'. We note also that C' has at least one non-zero entry in each row and in each of the first m columns, since it has rank m. Now write Y' for the matrix PYQ, which has $\frac{m(m+1)}{2}$ independent indeterminates in the positions (i,j) with $i \leqslant j$, and zeroes elsewhere, and write $\mathcal{Y}'$ for the space of completions of Y'. Then, as discussed in the proof of Proposition 3.5, every non-zero element of $\mathbb{F}^m$ arises as a basis for the rowspace of some element of $\mathcal{Y}'\backslash\mathcal{Y}\cap C'^*$ of rank 1. If $v_1 \in \mathbb{F}^m$ has a non-zero entry in its first position, then a completion of Y' can have $\langle v_1\rangle$ as its rowspace only if its non-zero entries are all in Row 1. It follows that an element of $\mathbb{F}^m$ whose first entry is not zero cannot be in the trace complement of the first column of C'. Then the first column of C' must have a unique non-zero entry, in its first position.

Thus the trace complement of Column 1 of C' is the hyperplane of $\mathbb{F}^m$ consisting of all elements whose first entry is zero. Let v_2 be an element of $\mathbb{F}^m$ whose first non-zero entry is in the second position. Then v_2 can occur only as the first or second row of a completion of Y', and it must span the rowspace of some completion Y_2 of Y' for which $\mathrm{trace}(Y_2C') \neq 0$. Thus v_2 cannot be in the trace complement of Column 2 of C'. It follows that the entry in the $(2,2)$ position of C' is not zero, and that all subsequent entries of Column 2 of C' *are* zeroes.

Now the trace complements of Columns 1 and 2 of C' in $\mathbb{F}^m$ intersect in the subspace of $\mathbb{F}^m$ consisting of all elements with zeroes in the first two positions; the argument above can be applied successively to Columns 3 through m of C', establishing the positions $(1,1),\ldots,(m,m)$ as the only locations of non-zero entries of C'. $\square$

This completes our proof of Theorem 3.3 : the partial matrix $A' = Q^{-1}AP^{-1}$ was obtained from A by row and column permutations and has the required form.

We now turn our attention to the more general Theorem 3.2, our characterization of rectangular partial matrices with the maximum possible number of indeterminates subject to a specific lower bound for the ranks of their completions. The strategy of our proof of Theorem 3.2 is to reduce the problem to the situation of Theorem 3.3, again using the duality discussed in Section 2. The principal tool needed to effect this reduction is the following lemma.

Lemma 3.8. *Suppose that $k \leqslant m \leqslant n$, and let R be a subspace of $M_{n\times m}(\mathbb{F})$ containing a subspace R_1 of codimension 1 such that $R\backslash R_1$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$. Furthermore suppose that R has the minimal possible dimension $\frac{k(k+1)}{2}$ subject to these conditions. Let $v \in (\mathbb{F}^m)^T$. Then the subspace $R(v) = \{Bv : B \in R\}$ of $(\mathbb{F}^n)^T$ can have dimension at most k.*

Proof. If $k = 1$, then R has dimension 1 and the statement is obviously true. So assume $k \geqslant 2$.

Let R_v denote the subspace of R consisting of all those elements whose right nullspace contains v. Since a complement of R_v in R must have dimension equal to that of $R(v)$, we have

$$\dim R = \frac{k(k+1)}{2} = \dim R_v + \dim R(v).$$

Let H be a complement of $\langle v\rangle$ in $(\mathbb{F}^m)^T$, so $(\mathbb{F}^m)^T = \langle v\rangle \oplus H$. Every $(k-1)$-dimensional subspace of $(\mathbb{F}^m)^T$ that contains v intersects H in a space of dimension $k-2$, and every subspace of H of dimension $k-2$ is contained in a unique $(k-1)$-dimensional subspace of $(\mathbb{F}^m)^T$ containing v. Since $R\backslash R_1$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$, every $(k-1)$-dimensional subspace of $(\mathbb{F}^m)^T$ that contains v, hence every $(k-2)$-dimensional subspace of H, is annihilated by an element of R_v that does not belong to R_1. Restricting to H, we may identify R_v with a space R^H of linear transformations from H to $(\mathbb{F}^n)^T$, and $R_1 \cap R_v$ with a hyperplane R_1^H of R^H, for which the set $R^H \backslash R_1^H$ has the dimension $(k-2)$ annihilation property on H. Then R_v must have dimension at least $\frac{(k-1)k}{2}$ by Theorem 2.10, and so

$$\dim R(v) = \frac{k(k+1)}{2} - \dim R_v \leqslant k$$

as required. $\square$

In the special case where R is the space of completions of a partial matrix $R^p \in P_{n\times m}(\mathbb{F})$ (whose constant part is the zero matrix), Lemma 3.8 means that all of the indeterminates in R^p are confined to at most k rows and to at most k columns. To see this in the case of the rows, suppose that $k+1$ distinct rows of R^p contain indeterminates: for convenience suppose these are the first $k+1$ rows. For any element $(a_1, a_2, \ldots, a_{k+1})$ of $\mathbb{F}^{k+1}$, let $R_{(a_1,a_2,\ldots,a_{k+1})}$ be the completion of R^p in which the first indeterminate in Row i is assigned the value a_i for $i = 1, \ldots, k+1$, and all other indeterminates are assigned the value 0. Let v be the element of $(\mathbb{F}^m)^T$ whose entries are all equal to 1. Then

$$R_{(a_1,a_2,\ldots,a_{k+1})}v = (a_1, \ldots, a_{k+1}, 0, \ldots, 0)^T \in (\mathbb{F}^n)^T,$$

and so $R(v)$ has dimension at least $k+1$, contrary to Lemma 3.8.

That the indeterminates of R^p are also confined to k columns is now an immediate consequence of Lemma 2.11, since $R^T \backslash R_1^T$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^n)^T$, and has minimum possible dimension $\frac{k(k+1)}{2}$ subject to this condition.

We are now in a position to prove Theorem 3.2. As in the proof of Theorem 3.3, we write C and X for the constant and indeterminate parts of our partial matrix $A \in P_{m\times n}(\mathbb{F})$, $\mathcal{X}$ for the space of completions of X, $\mathcal{Y}$ for the trace complement of $\mathcal{X}$ in $M_{n\times m}(\mathbb{F})$, and Y for the partial matrix in $P_{n\times m}(\mathbb{F})$ of which $\mathcal{Y}$ is the space of completions.

Proposition 3.9. *Let k be a positive integer, $k \leqslant m$. Let $A \in P_{m\times n}(\mathbb{F})$ be a partial matrix with $mn - \frac{k(k+1)}{2}$ indeterminates, all of whose completions have rank at least k. Then A may be transformed by a row and column permutation to a partial matrix in which all constant entries occur in the first k rows and in the first k columns.*

Proof. By Theorem 2.9, $\mathcal{Y} \backslash \mathcal{Y} \cap C^*$ has the dimension $(k-1)$ annihilation property on $(\mathbb{F}^m)^T$. By Lemma 3.8 and the remarks following it, the $\frac{k(k+1)}{2}$ indeterminates of Y collectively occupy at most k rows and at most k columns. Thus there exist permutation matrices $P \in M_n(\mathbb{F})$ and $Q \in M_m(\mathbb{F})$ for which PYQ is fully occupied by zeroes outside its upper left $k \times k$ region, and for which $Q^{-1}XP^{-1}$ is fully occupied by indeterminates outside its upper left $k \times k$ region. $\square$

The concluding step in the proof of Theorem 3.2 is a direct application of Theorem 3.3. The matrix $A' = Q^{-1}AP^{-1}$ is fully occupied by indeterminates outside its upper left $k \times k$ region, which contains $\frac{k(k-1)}{2}$ indeterminates and $\frac{k(k+1)}{2}$ constants. Let $A'_k \in P_k(\mathbb{F})$ denote the partial $k \times k$ matrix whose entries are those of the upper left $k \times k$ region of A'. Every completion of A' in which all entries outside this region are zeroes has rank at least k, hence exactly k. Thus A'_k is a partial matrix with $\frac{k(k-1)}{2}$ indeterminates in $P_k(\mathbb{F})$, whose completions all have rank k. By Theorem 3.3, A'_k can be transformed by row and column permutations to an upper triangular matrix in which the entries on the main diagonal are non-zero constants and the entries above the main diagonal are indeterminates. Since these permutations can be extended to permutations of the rows and columns of A' that affect only the first k rows and first k columns, our proof of Theorem 3.2, which is restated below, is complete.

Theorem 3.2. Let $\mathbb{F}$ be a field and let $A \in P_{m\times n}(\mathbb{F})$ be a partial matrix with the property that every $\mathbb{F}$-completion of A has rank at least k for some fixed $k \leqslant \min(m, n)$. Then the number of indeterminates in A is at most $mn - \frac{k(k+1)}{2}$. This bound is attained if and only if A may be transformed by row and column permutations to a partial matrix A' of the following form:

- The upper left $k \times k$ submatrix of A' is an upper triangular matrix with non-zero constant entries on the main diagonal and independent indeterminates above the main diagonal.
- All entries of A' outside the upper left $k \times k$ region are independent indeterminates.

Acknowledgments

The authors are grateful to James Cruickshank for valuable discussions related to the development of the ideas in this article.

References

[1] R. Brualdi, Z. Huang, X. Zhan, Singular, non-singular and bounded rank completions of ACI-matrices, Linear Algebra Appl. 433 (2010) 1452–1462.
[2] N. Cohen, C.R. Johnson, L. Rodman, H. Woerdeman, Ranks of completions of partial matrices, Oper. Theory Adv. Appl. 40 (1989) 165–185.
[3] C. de Seguins Pazzis, On the matrices of given rank in a large subspace, Linear Algebra Appl. 435 (2011) 147–151.
[4] M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices, I, Amer. J. Math. 80 (1958) 614–622.
[5] D.J. Hartfiel, R. Loewy, A determinantal version of the Frobenius–König theorem, Linear and Multilinear Algebra 16 (1984) 155–165.
[6] Z. Huang, X. Zhan, ACI-matrices all of whose completions have the same rank, Linear Algebra Appl. 434 (2011) 1956–1967.
[7] R. Meshulam, On two extremal matrix problems, Linear Algebra Appl. 114/115 (1989) 261–271.
[8] R. Quinlan, Spaces of matrices without non-zero eigenvalues in their field of definition, and a question of Szechtman, Linear Algebra Appl. 434 (2011) 1580–1587.